

- Definition 4.10: Domain, Range, and Field

$$\text{dom}R = \{x \mid \exists y (\langle x, y \rangle \in R)\}$$

$$\text{ran}R = \{y \mid \exists x (\langle x, y \rangle \in R)\}$$

$$\text{fld}R = \text{dom}R \cup \text{ran}R$$

e.g. >>> Example:

$$R = \{\langle a, \{b\} \rangle, \langle c, d \rangle, \langle \{a\}, \{d\} \rangle, \langle d, \{d\} \rangle\}, \text{ then}$$

$$\text{dom}R = \{a, c, \{a\}, d\}$$

$$\text{ran}R = \{\{b\}, d, \{d\}\}$$

$$\text{fld}R = \{a, c, \{a\}, d, \{b\}, \{d\}\}$$

↳ Inverse and Composition of Relations

- Definition 4.11: The *inverse* of R

$$R^{-1} = \{ \langle y, x \rangle \mid \langle x, y \rangle \in R \}$$

- Definition 4.12: *Composition* of R and S

$$R \circ S = \{ \langle x, z \rangle \mid \exists y (\langle x, y \rangle \in R \wedge \langle y, z \rangle \in S) \}$$

e.g. >>> Example: $R = \{ \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 2 \rangle \}$

$$S = \{ \langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \}$$

$$R^{-1} = \{ \langle 2, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle, \langle 2, 2 \rangle \}$$

$$R \circ S = \{ \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle \}$$

$$S \circ R = \{ \langle 1, 2 \rangle, \langle 1, 4 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \}$$

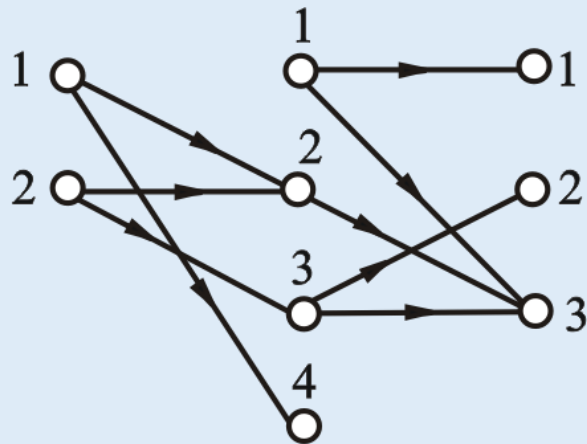
4.2.1 Basic Operations of Relations

↳ Find the composition using a diagram

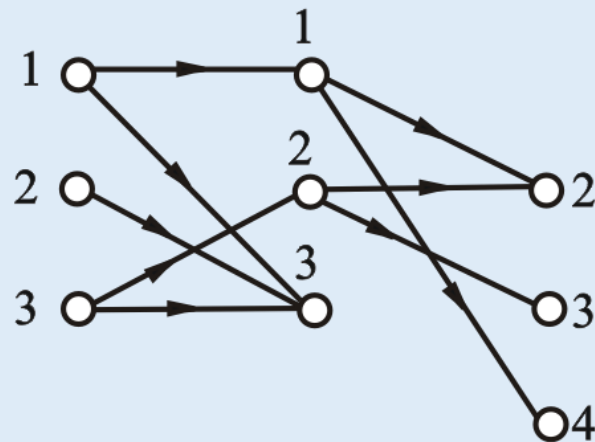
Use the diagrammatic (not relational diagram) method to find the composition.

e.g. Example: $R = \{ \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 2 \rangle \}$

$S = \{ \langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \}$



$R \circ S$



$S \circ R$

$$R \circ S = \{ \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle \}$$

$$S \circ R = \{ \langle 1, 2 \rangle, \langle 1, 4 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \}$$

↳ Properties of Inverse Operations

■ **Theorem 4.1:** Let F be an arbitrary relation, then:

(1) $(F^{-1})^{-1} = F$

(2) $\text{dom}F^{-1} = \text{ran}F$, $\text{ran}F^{-1} = \text{dom}F$

■ **Proof:**

(1) For any $\langle x, y \rangle$, by the definition of inverse, we have

$$\langle x, y \rangle \in (F^{-1})^{-1} \Leftrightarrow \langle y, x \rangle \in F^{-1} \Leftrightarrow \langle x, y \rangle \in F$$

Thus, $(F^{-1})^{-1} = F$

(2) For any x ,

$$x \in \text{dom}F^{-1} \Leftrightarrow \exists y (\langle x, y \rangle \in F^{-1})$$

$$\Leftrightarrow \exists y (\langle y, x \rangle \in F) \Leftrightarrow x \in \text{ran}F$$

Thus, $\text{dom}F^{-1} = \text{ran}F$.

Similarly, we can prove $\text{ran}F^{-1} = \text{dom}F$.

↳ Associativity and Inverse Operations of Relational Composition

■ Theorem 4.2:

Let F , G , and H be arbitrary relations, then:

$$(1) (F \circ G) \circ H = F \circ (G \circ H)$$

$$(2) (F \circ G)^{-1} = G^{-1} \circ F^{-1}$$

■ Proof (1) : For any $\langle x, y \rangle$,

$$\langle x, y \rangle \in (F \circ G) \circ H$$

$$\Leftrightarrow \exists t (\langle x, t \rangle \in F \circ G \wedge \langle t, y \rangle \in H)$$

$$\Leftrightarrow \exists t (\exists s (\langle x, s \rangle \in F \wedge \langle s, t \rangle \in G) \wedge \langle t, y \rangle \in H)$$

$$\Leftrightarrow \exists t \exists s (\langle x, s \rangle \in F \wedge \langle s, t \rangle \in G \wedge \langle t, y \rangle \in H)$$

$$\Leftrightarrow \exists s (\langle x, s \rangle \in F \wedge \exists t (\langle s, t \rangle \in G \wedge \langle t, y \rangle \in H))$$

$$\Leftrightarrow \exists s (\langle x, s \rangle \in F \wedge \langle s, y \rangle \in G \circ H)$$

$$\Leftrightarrow \langle x, y \rangle \in F \circ (G \circ H)$$

Thus, $(F \circ G) \circ H = F \circ (G \circ H)$

↳ Associativity and Inverse Operations of Relational Composition

■ Theorem 4.2:

Let F , G , and H be arbitrary relations, then:

$$(1) (F \circ G) \circ H = F \circ (G \circ H)$$

$$(2) (F \circ G)^{-1} = G^{-1} \circ F^{-1}$$

■ Proof (2): For any $\langle x, y \rangle$,

$$\langle x, y \rangle \in (F \circ G)^{-1}$$

$$\Leftrightarrow \langle y, x \rangle \in F \circ G$$

$$\Leftrightarrow \exists t (\langle y, t \rangle \in F \wedge \langle t, x \rangle \in G)$$

$$\Leftrightarrow \exists t (\langle x, t \rangle \in G^{-1} \wedge \langle t, y \rangle \in F^{-1})$$

$$\Leftrightarrow \langle x, y \rangle \in G^{-1} \circ F^{-1}$$

$$\text{Thus } (F \circ G)^{-1} = G^{-1} \circ F^{-1}$$

↳ Relational Composition of the Identity Relation I_A

■ Theorem 4.3: Composition of the Identity Relation I_A

Let R be a relation on A , I_A is the Identity Relation on Set A

then $R \circ I_A = I_A \circ R = R$

■ Proof For any pair $\langle x, y \rangle$

$$\langle x, y \rangle \in R \circ I_A$$

$$\Leftrightarrow \exists t (\langle x, t \rangle \in R \wedge \langle t, y \rangle \in I_A)$$

$$\Leftrightarrow \exists t (\langle x, t \rangle \in R \wedge t = y \wedge y \in A)$$

$$\Leftrightarrow \langle x, y \rangle \in R$$

Thus, $R \circ I_A = R$.

Similarly, we can prove that $I_A \circ R = R$.

■ Definition 4.13: R^n

Let R be a relation on A , and n be a natural number. The n -th power of R is defined as:

$$(1) R^0 = \{ \langle x, x \rangle \mid x \in A \} = I_A$$

$$(2) R^{n+1} = R^n \circ R$$

Note:

- For any relations R_1 and R_2 on A , we have,

$$R_1^0 = R_2^0 = I_A$$

- For any relation R on A , we have: $R^1 = R$

- For a relation R represented by a set, computing R^n means the composition of R with itself n times.
 - The n -th power of a relation is equal to the n -th power of its matrix representation.
 - The matrix representation of a relation is obtained by matrix multiplication, where addition is performed using logical addition.
- e.g.* >>> **Example:** Let $A = \{a, b, c, d\}$, $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$,
Find the powers of R , and represent them using both a matrix and a relation diagram.

Solution: The relation matrix for R and R^2 are as follows:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e.g. >>> **Example:** Let $A = \{a, b, c, d\}$, $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$,
Find the powers of R , and represent them using both a matrix and a relation diagram.

Solution: The relation matrix for R^3 and R^4 are as follows:

$$M^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus: $M^4 = M^2$, $R^4 = R^2$. Then we can find

$$R^2 = R^4 = R^6 = \dots, \quad R^3 = R^5 = R^7 = \dots$$

$$R^0 = I_A \text{ relation matrix : } M^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ Methods of power operations • relation diagram

Let R be a relation on $A = \{a, b, c, d\}$,

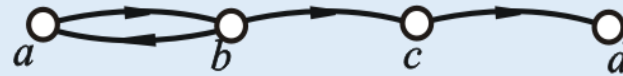
$R = \{ \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle \}$,

Using the relation diagram method, the relation diagrams

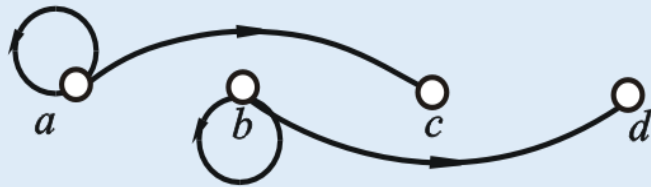
for R^0, R^1, R^2, R^3



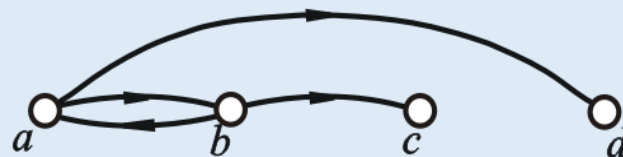
R^0



$R^1=R$



$R^2=R^4$



$R^3=R^5$

4.2.2 Power Operations of Relations • The properties of a power operations

↳ periodicity or eventual stability

- **Theorem 4.4:** The periodicity or eventual stability of a power operation under finite exponents.

Let A be a set with n elements, and let R be a relation on A .

Then, there exist natural numbers s and t such that $R^s = R^t$.

- **Proof Outline:**

(1) A relation R on A is a subset of $A \times A$, which contains 2^{n^2} at most pairs.

(2) Since each R^s is a subset of $A \times A$, there are at most 2^{n^2} distinct possible relations.

(3) The sequence $R: R^0, R^1, R^2, R^3, \dots$ has infinitely many indices but only finitely many distinct relations 2^{n^2} , so by the pigeonhole principle, there exist $s \neq t$ such that $R^s = R^t$.

↳ Composition of Powers and Power of a Power properties

- Theorem 4.5: *Composition of Powers and Power of a Power* properties.

Let R be a relation on A , and $m, n \in \mathbb{N}$, Then

$$(1) R^m \circ R^n = R^{m+n}$$

$$(2) (R^m)^n = R^{mn}$$

- Proof: By induction.

(1) For any given $m \in \mathbb{N}$, , we induct on n .

If $n=0$, then

$$R^m \circ R^0 = R^m \circ I_A = R^m = R^{m+0}$$

Assume $R^m \circ R^n = R^{m+n}$, then

$$R^m \circ R^{n+1} = R^m \circ (R^n \circ R) = (R^m \circ R^n) \circ R = R^{m+n+1},$$

Thus, for all $m, n \in \mathbb{N}$ we have $R^m \circ R^n = R^{m+n}$.

↳ Composition of Powers and Power of a Power properties

- Theorem 4.5: *Composition of Powers and Power of a Power* properties.

Let R be a relation on A , and $m, n \in \mathbb{N}$, Then

$$(1) \quad R^m \circ R^n = R^{m+n}$$

$$(2) \quad (R^m)^n = R^{mn}$$

- Proof: By induction.

(2) For any given $m \in \mathbb{N}$, we induct on n .

If $n = 0$, then

$$(R^m)^0 = I_A = R^0 = R^{m \times 0}$$

Assume $(R^m)^n = R^{mn}$, then

$$(R^m)^{n+1} = (R^m)^n \circ R^m = (R^{mn}) \circ R^m = R^{mn+m} = R^{m(n+1)}$$

Thus, for any $m, n \in \mathbb{N}$ $(R^m)^n = R^{mn}$.

↳ Stabilization, Periodicity and Finite State Constraint

■ **Theorem 4.6:**, Stabilization, Periodicity and Finite State Constraint in the Powers of a Relation

Let R be a relation on A . If there exist natural numbers s, t ($s < t$) such that $R^s = R^t$, then

(1) For any $k \in \mathbb{N}$, $R^{s+k} = R^{t+k}$ (*Stabilization Property*)

The two powers are equal and remain unchanged when the same power is added.

(2) For any $k, i \in \mathbb{N}$ $R^{s+kp+i} = R^{s+i}$, where $p = t-s$ (*Periodicity Property*)

The period for the equality of the two powers is p

(3) Let $S = \{R^0, R^1, \dots, R^{t-1}\}$, then for any $q \in \mathbb{N}$, $R^q \in S$ (*Finite State Constraint*)

The natural number powers of a relation R on a finite set always have a period.

■ Proof:

$$(1) R^{s+k} = R^s \circ R^k = R^t \circ R^k = R^{t+k}$$

(2) Induct on k . If $k=0$, then we have

$$R^{s+0p+i} = R^{s+i}$$

Assume $R^{s+kp+i} = R^{s+i}$, where $p = t-s$, then

$$\begin{aligned} R^{s+(k+1)p+i} &= R^{s+kp+i+p} = R^{s+kp+i} \circ R^p \\ &= R^{s+i} \circ R^p = R^{s+p+i} = R^{s+t-s+i} = R^{t+i} = R^{s+i} \end{aligned}$$

By the principle of mathematical induction, the proposition is proven.

■ Proof:

(3) For any $q \in \mathbb{N}$, if $q < t$, it is obvious that $R^q \in S$.

If $q \geq t$, then there exist natural numbers k and i such that

$$q = s + kp + i, \text{ where } 0 \leq i \leq p - 1.$$

Thus,

$$R^q = R^{s+kp+i} = R^{s+i}$$

Since $s + i \leq s + p - 1 = s + t - s - 1 = t - 1$

this proves that $R^q \in S$.

4.2 Relational Operation • Brief summary

Objective :

Key Concepts :



离散数学 (011122)



魏可佶

kejiwei@tongji.edu.cn
<https://kejiwei.github.io/>



- 4.1 Definition and Representation of Relations
- 4.2 Relational Operations
- **4.3 Properties of Relations**
- **4.4 Equivalence Relations and Partial Order Relations**

■ 4.3.1 Definition and Determination of Relation Properties

- Reflexivity and Irreflexivity
- Symmetry and Antisymmetry
- Transitivity

■ 4.3.2 Closure of Relations

- Definition of Closure
- Closure Calculation
- Warshall's Algorithm

↳ Reflexivity and Irreflexivity

■ Definition 4.14: Reflexivity and irreflexivity of Relations

Let R be a relation on A .

(1) If $\forall x(x \in A \rightarrow \langle x, x \rangle \in R)$, then R is called *reflexive* on A .

(2) If $\forall x(x \in A \rightarrow \langle x, x \rangle \notin R)$, then R is called *irreflexive* on A .

■ **Reflexive:** The universal relation E_A on A , the identity relation I_A , the less-than-or-equal relation L_A , and the divisibility relation D_A .

■ **Irreflexive:** The less-than relation ($<$) on the real number set and the strict inclusion relation (\subsetneq) on the power set.

↳ Reflexivity and Irreflexivity(e.g.)

e.g. >>> **Example:** $A = \{a, b, c\}$, R_1, R_2, R_3 is the relation on A , where

$$R_1 = \{ \langle a, a \rangle, \langle b, b \rangle \}, R_2 = \{ \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle a, b \rangle \}, R_3 = \{ \langle a, c \rangle \}$$

To determine whether the relations R_1, R_2, R_3 are reflexive or irreflexive.

■ Reflexivity check:

- Since $(c, c) \notin R_1$, R_1 is not reflexive.
- All of $(a, a), (b, b), (c, c)$ are in R_2 , R_2 is reflexive.
- None of $(a, a), (b, b), (c, c)$ are present, R_3 is not reflexive.

■ Irreflexivity check:

- R_1 contains (a, a) and (b, b) , meaning some elements have self-loops, R_1 is not irreflexive.
- Since R_2 contains self-loops $((a, a), (b, b), (c, c))$, R_2 is not irreflexive.
- R_3 does not contain any self-loops $((x, x))$, R_3 is irreflexive.

■ Final Summary: R_1 : Neither reflexive nor irreflexive; R_2 : Reflexive;
 R_3 : Irreflexive

↳ Symmetric & Antisymmetric

■ Definition 4.15: Symmetric and Antisymmetric of Relations.

Let R be a relation on A ,

(1) If $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R)$, then R is called a *symmetric relation* on A .

(2) If $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \rightarrow x = y)$, then R is called an *antisymmetric relation* on A .

■ Such as:

Symmetric: The universal E_A on A , the identity relation I_A , and the empty relation \emptyset .

Antisymmetric: The identity relation I_A and the empty relation \emptyset are antisymmetric relations on A .

📄 **Note:** The formulas (1) and (2) iterates over all elements x, y in A , but the actual constraint applies only to the elements in R .

↳ Symmetric & Antisymmetric (e.g.)

e.g. >>> Example 2: Let $A = \{a, b, c\}$, and R_1, R_2, R_3 and R_4 are relations on A , where...

$$R_1 = \{\langle a, a \rangle, \langle b, b \rangle\}, \quad R_2 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle\}$$

$$R_3 = \{\langle a, b \rangle, \langle a, c \rangle\}, \quad R_4 = \{\langle a, b \rangle, \langle b, a \rangle, \langle a, c \rangle\}$$

To determine whether the relations R_1, R_2, R_3, R_4 are Symmetric or Antisymmetric.

	Symmetric check: $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R)$	Antisymmetric check: $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \rightarrow x = y)$	Conclusion
R_1	R_1 only contains (a,a) and (b,b).	Only contains reflexive elements (x,x) and does not include any (x,y) such that $x \neq y$.	Symmetric: <input checked="" type="checkbox"/> Yes Antisymmetric: <input checked="" type="checkbox"/> Yes
R_2	Every $(x, y) \in R_2$, the corresponding (y, x) is also in R_2 .	Contains (a,b) and (b,a), but $a \neq b$, which violates the requirement that $x = y$.	Symmetric: <input checked="" type="checkbox"/> Yes Antisymmetric: <input checked="" type="checkbox"/> No

↳ Symmetric & Antisymmetric (e.g.)

e.g. >>> **Example:** Let $A = \{a, b, c\}$, and R_1, R_2, R_3 and R_4 are relations on A , where...

$$R_1 = \{\langle a, a \rangle, \langle b, b \rangle\}, \quad R_2 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle\}$$

$$R_3 = \{\langle a, b \rangle, \langle a, c \rangle\}, \quad R_4 = \{\langle a, b \rangle, \langle b, a \rangle, \langle a, c \rangle\}$$

To determine whether the relations R_1, R_2, R_3, R_4 are Symmetric or Antisymmetric.

	Symmetric check: $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R)$	Antisymmetric check: $\forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \rightarrow x = y)$	Conclusion
R_3	$(a, b) \in R_3$ but $(b, a) \notin R_3$, and $(a, c) \in R_3$ but $(c, a) \notin R_3$	no pairs (x, y) and (y, x) in R_3 for $x \neq y$, making the condition trivially true.	Symmetric: ✗ No Antisymmetric: <input checked="" type="checkbox"/> Yes
R_4	$(a, c) \in R_4$ but $(c, a) \notin R_4$	$(a, b) \in R_4$ and $(b, a) \in R_4$, but $a \neq b$	Symmetric: ✗ No Antisymmetric: ✗ No

↳ Transitive relation

- Definition 4.16: transitive relation on A .

Let R be a relation on A . If

$$\forall x \forall y \forall z (x, y, z \in A \wedge \langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \rightarrow \langle x, z \rangle \in R),$$

then R is called a *transitive relation* on A .

- **Such as:** The universal relation E_A on A , the identity relation I_A , the empty relation \emptyset , the less-than-or-equal relation, the less-than relation, the divisibility relation, the inclusion relation, and the strict inclusion relation.

↳ Transitive relation (e.g.)

e.g. >>> Example: Let $A = \{a, b, c\}$, R_1, R_2, R_3 relation on A , where
 $R_1 = \{\langle a, a \rangle, \langle b, b \rangle\}$; $R_2 = \{\langle a, b \rangle, \langle b, c \rangle\}$; $R_3 = \{\langle a, c \rangle\}$
 To determine whether the relations R_1, R_2, R_3 are Transitive relation on A .

	$\forall x \forall y \forall z (x, y, z \in A \wedge \langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \rightarrow \langle x, z \rangle \in R)$	Conclusion
R_1	only contains reflexive elements and has no pairs that could violate transitivity.	Transitive relation <input checked="" type="checkbox"/> Yes
R_2	because $(a, b) \in R_2$ and $(b, c) \in R_2$, but $(a, c) \notin R_2$	Transitive relation <input type="checkbox"/> No
R_3	contains only a single pair and has no chains to check for transitivity violations.	Transitive relation <input checked="" type="checkbox"/> Yes